Let us now come to the study of non-linear equations.

<u>Definition 9.6</u> (separable eqs.):

A first order ODE $y' = f(x,y)$ is said to be separable if $F(x,y)$ can be expressed as a product

$$y' = g(x)\,h(y), \quad x \in I. \qquad (1)$$

<u>Solution Method</u>:

Assume that the solution $y = y(x)$ exists, and that $h(y) \neq 0$ for $x \in I$. Then dividing by $h(y)$ gives

$$\frac{y'(x)}{h[y(x)]} = g(x).$$

Integrating and using the substitution $dy = y'dx$ then gives

$$\int \frac{y'(x)}{h[y(x)]}\,dx = \int \frac{dy}{h(y)} = H(y)$$

Hence we arrive at

$$H[y(x)] = \int g(x)\,dx + C \qquad (2)$$

where $H(y)$ is a function with the property:

$$H'(y) = \frac{1}{h(y)}.$$

$\longrightarrow$ Equation (2) gives an implicit form of the solution

<u>Example 9.6:</u>

i) $y' = \dfrac{x-5}{y^2}$

solution: We multiply both sides with y^2 and get

$$y^2 y' = (x-5)$$

Integrating both sides we get

$$y^3/3 = x^2/2 - 5x + C.$$

Hence, $$y = \left[3x^2/2 - 15x + C\right]^{\frac{1}{3}}.$$

ii) $y' = \dfrac{y-1}{x+3}$ $(x > -3)$.

solution: By inspection, $y=1$ is a solution. Dividing both sides of the given eq. by $y-1$ we get

$$\frac{y'}{y-1} = \frac{1}{x+3}$$

(possible if $y(x) \neq 0$)

Integrating both sides we get

$$\int \frac{y'}{y-1}\,dx = \int \frac{dx}{x+3} + C_1,$$

from which we get

$$\log|y-1| = \log|x+3| + C_1.$$

Thus

$$|y-1| = e^{C_1}(x+3),$$

from which by solving for y and letting $C := \pm e^{C_1}$ we get

$$y = 1 + C(x+3)$$

iii) $y' = \dfrac{y\cos x}{1+2y^2}$

solution: Multiplying both sides by $\dfrac{1+2y^2}{y}$ and integrating gives

$$\int \frac{(1+2y^2)}{y}\,dy = \int \cos x\,dx + C,$$

from which we get a family of solutions

$$\log|y| + y^2 = \sin x + C,$$

where C is an arbitrary constant.

This is not the most general solution

as does not contain the solution $y=0$!
With initial condition $y(0)=1$, we get $C=1$, hence the solution

$$\log|y| + y^2 = \sin x + 1.$$

Example 9.7 (Logistic equation):

$$y' = ay(b-y), \quad a,b>0 \text{ fixed constants}$$

(Recall $P(t) = kP(1-\frac{P}{M}) = \frac{k}{M}P(M-P)$
$\Rightarrow a = \frac{k}{M}, \ b = M$)

We already know about the two constant solutions $y=0$ and $y=b$

To find more general solutions, we rewrite the logistic eq. as:

$$\frac{y'}{y(b-y)} = a$$

Integrating both sides we get

$$\int \frac{y'dt}{y(b-y)} = at + C$$

$$\text{or} \quad \int \frac{dy}{y(b-y)} = at + C. \qquad (*)$$

By partial fractions

$$\frac{1}{Y(b-Y)} = \frac{1}{b}\left(\frac{1}{Y} + \frac{1}{b-Y}\right)$$

$\rightarrow$ (*) can be written as

$$\frac{1}{b}\left(\log|Y| - \log|b-Y|\right) = at + C.$$

Multiplying both sides by b and exponentiating gives

$$\frac{|Y|}{|b-Y|} = e^{bC} e^{abt} = C_1 e^{abt},$$

where the arbitrary constant $C_1 = e^{bC} > 0$ can be determined by the initial condition:

$Y(0) = Y_0$ as $\quad C_1 = \dfrac{|Y_0|}{|b-Y_0|}$

Two cases need to be discussed separately.

<u>Case I:</u> $Y_0 < b$: one has $C_1 = \left|\frac{Y_0}{b-Y_0}\right| = \frac{Y_0}{b-Y_0} > 0.$

So $$\frac{|Y|}{|b-Y|} = \left(\frac{Y_0}{b-Y_0}\right) e^{abt} > 0, \quad t \in \underline{I}$$

From the above we derive

$$\frac{Y}{b-Y} = C_1 e^{abt}, \quad Y = (b-Y) C_1 e^{abt}$$

$$\Rightarrow \quad Y = \frac{b C_1 e^{abt}}{1 + C_1 e^{abt}}$$

This shows that if $Y_0 = 0$, one has the solution $Y(t) = 0$. However, if $0 < Y_0 < b$, one has the solution $0 < Y(t) < b$, and as $t \to \infty$, $Y(t) \to b$.

<u>Case II:</u> $Y_0 > b$

One has $C_1 = \left| \frac{Y_0}{b - Y_0} \right| = \frac{-Y_0}{b - Y_0} > 0$. Then

$$\left| \frac{Y}{b-Y} \right| = \left(\frac{Y_0}{Y_0 - b} \right) e^{abt} > 0, \quad t \in I$$

From this we get

$$\frac{Y}{Y-b} = \left(\frac{Y_0}{Y_0 - b} \right) e^{abt}$$

$$\Leftrightarrow \quad Y = \frac{b \left(\frac{Y_0}{Y_0 - b} \right) e^{abt}}{\left(\frac{Y_0}{Y_0 - b} \right) e^{abt} - 1} .$$

It shows that if $Y_0 > b$, one has the solution $Y(t) > b$, and as $t \to \infty$, $Y(t) \to b$.

Altogether, Case I and Case II show:

- $Y(t) = 0$ is an unstable equilibrium of the system
- $Y(t) = b$ is a stable equilibrium of the system.

§ 9.4 Higher order Equations:

The most general linear second-order differential equation is given by

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \quad (1)$$

where P, Q, R and G are continuous functions of x.

Definition 9.7:

The case $G(x)=0$ in (1) is called a "homogeneous" second order linear ODE.
If $G(x) \neq 0$, eq. (1) is called inhomogeneous.

Proposition 9.3:

If $y_1(x)$ and $y_2(x)$ are both solutions of the homogeneous eq. $P(x)y'' + Q(x)y' + R(x)y = 0$, (2) then the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad c_1, c_2 \in \mathbb{R}$$

is also a solution of (2).

Proof: $P(x)y'' + Q(x)y' + R(x)y$

$$= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2)$$

$$= c_1\left[P(x)y_1'' + Q(x)y_1' + R(x)y_1\right] + c_2\left[P(x)y_2'' + Q(x)y_2' + R(x)y_2\right] = 0$$

$\square$

Solutions are in general not easily discovered
$\longrightarrow$ simplify by taking P, R, Q to be constants:

$$a y'' + b y' + c y = 0 \qquad (3)$$

for some constants a, b, c in $\mathbb{R}$.
Substitute the ansatz $y = e^{rx}$ into (3):

$$a r^2 e^{rx} + b r e^{rx} + c e^{rx} = 0$$

or $$(a r^2 + b r + c) e^{rx} = 0$$

Since $e^{rx} \neq 0 \;\; \forall\, x \in \mathbb{R}$, $y = e^{rx}$ is a solution of (2) if r is a root of the equation:

$$a r^2 + b r + c = 0 \qquad (4)$$

$\longrightarrow$ "characteristic equation"

Roots of (4) are given by the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$:

<u>1)</u>: $b^2 - 4ac > 0$ <u>2)</u>: $b^2 - 4ac = 0$ <u>3)</u>: $b^2 - 4ac < 0$

1): $b^2 - 4ac > 0$

In this case roots r_1 and r_2 are real and distinct, so $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions (meaning $y_1 \neq k y_2$ for any $k \in \mathbb{R}$)

$\Longrightarrow$ By Prop. 9.3

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \qquad (5)$$

is also a solution of (3)